Broadband Particle Filtering in a Noisy Littoral Ocean

J. V. Candy

February 24, 2015

Disclaimer

This document was prepared as an account of work sponsored by an agency of the United States government. Neither the United States government nor Lawrence Livermore National Security, LLC, nor any of their employees makes any warranty, expressed or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States government or Lawrence Livermore National Security, LLC. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States government or Lawrence Livermore National Security, LLC, and shall not be used for advertising or product endorsement purposes.

Broadband Particle Filtering in a Noisy Littoral Ocean

J. V. Candy

Fellow, IEEE

Lawrence Livermore National Laboratory, Livermore, CA 94551

Abstract—When sound propagates in the shallow ocean, source characteristics complicate the analysis of received acoustic data considerably especially when they are broadband and spatially complex. Noise whether ambient, distant shipping, wind blown surface generated complicates this already chaotic environment even further primarily because these disturbances propagate through the same inherent oceanic medium. The broadband problem can be decomposed into a set of narrowband problems by decomposing the source spectrum into its set of narrowband lines. A generic Bayesian solution to the broadband pressure-field enhancement and modal function extraction problem is developed that leads to a so-called nonparametric estimate of the desired posterior distribution enabling statistical inference.

Index Terms—littoral region, normal-modes, broadband Bayesian processor, sequential Monte Carlo, particle filter

I. INTRODUCTION

The characterization of transient sources has long been a problem with keen interest in the ocean acoustic community primarily because most targets are broadband. Initially, the work of many [1]-[9] began with a simple temporal matched-filter approach where a known source location was used to provide a replica employed to match to the measured data. An alternative to the temporal matched-filter approach evolved as a natural extension to model-based, matched-field/matched mode processing entailing the matching of a predicted to a measured pressure-field under an assumed source location. Matched-mode processing of broadband source signals also offers an improved enhancement capable of not only localizing the transient target, but also capable of estimating its broadband temporal spectrum [7]-[9].

Uncertainty of the ocean environment motivates the use of stochastic models to capture the random nature of the phenomena ranging from ambient noise and scattering to distant shipping and the nonstationary nature of this hostile medium. Therefore, processors that do not take these effects into account are doomed to large errors [10]-[14]. When contemplating the broadband problem it is quite natural to develop temporal techniques especially if the underlying model is the full wave equation; however, if we assume a normal-mode propagation model then it is more natural to: (1) filter the broadband receiver outputs into narrow bands; (2) process each band with a devoted processor; and then (3) combine the narrowband results to create a broadband solution.

The methodology employed is based on a state-space representation of the normal-mode propagation model as before [11]. It has already been shown that the state-space representation can be utilized for signal enhancement to spatially propagate modal and range functions in both the narrowband and broadband cases [11], [12]. In the random case, a stochastic model evolves allowing the inclusion of uncertain phenomena such as noise and modeling errors in a consistent manner [11], [12]. In our case, the measurement noise can be lumped to represent the near-field acoustic noise field, flow noise on the hydrophones and electronic noise, whereas the modal/range uncertainty can also be lumped to represent sound speed profile errors, spatially correlated noise from distant shipping, errors in the boundary conditions, sea state effects and ocean inhomogeneities [15].

We develop a generic Bayesian approach to solve the broadband enhancement problem primarily because it can easily be extended to solve a wide variety of general ocean acoustic problems that are not restricted to additive unimodal stochastic processes. This approach leads directly to the so-called *particle filter* (*PF*) that is a sequential Markov chain Monte Carlo (*MCMC*) Bayesian processor capable of providing reasonable performance for a multi-modal problem estimating a nonparametric representation of the posterior distribution [16]-[18].

We provide a brief discussion of the broadband problem in Sec. II and develop the underlying state-space model structures. Next in Sec. III, we develop the broadband Bayesian processor showing that it provides a generic Bayesian solution to the signal enhancement problem and apply it to noisy data synthesized from a littoral region. A shallow ocean simulation is developed in Sec. IV illustrating the construction of the processor as well as its performance when applied to synthesized noisy broadband pressure-field data. We summarize our results in the final section and discuss future work.

II. BROADBAND STATE-SPACE PROPAGATORS

In this section we summarize the development of a broadband (state-space) propagator. This propagator is embedded in the Bayesian particle filter solution. Next we briefly discuss the broadband normal-mode extension and then the transformation of this underlying model to state-space form.

A. Broadband Normal-Modes

It is well-known [2], [8], [13] in ocean acoustics that the pressure-field solution to the Helmholtz equation under the appropriate assumptions can be expressed as the sum of

normal modes. The Fourier transform of the pressure-field gives

$$p(r, z, \omega_o) = \sum_{m=1}^{M} a \mathcal{H}_o \left(\kappa_r(m) r \right) \phi_m(z_s) \phi_m(z) \delta(\omega - \omega_o) \quad (1)$$

indicating a *narrowband* solution or equivalently a line at ω_o in the temporal frequency domain.

This modal representation has been extended to include a broadband source, $s(t)$, with corresponding spectrum, $S_s(\omega)$ by [8]-[10]. In this case, the ocean medium is specified by its *Green's* function (impulse response) which can be expressed in terms of the inherent normal modes spanning the water column

$$\mathcal{G}(r, z, \omega) := \sum_m \mathcal{H}_o \left(\kappa_r(m) r \right) \phi_m(z_s, \omega) \phi_m(z, \omega) \quad (2)$$

Here r and z_s are the source range and depth coordinates, respectively and $\phi_m(z, \omega)$ is the mth modal or eigensolution, satisfying the *vertical* or *depth* equation obtained through separation of variables applied to the Helmholtz equation [2] given by

$$\frac{d^2}{dz^2} \phi_m(z, \omega) + \kappa_z^2(m, \omega) \phi_m(z, \omega) = 0. \quad (3)$$

The wave numbers satisfy the corresponding *dispersion relation*

$$\frac{\omega^2}{c^2(z)} = \kappa_r^2(m, \omega) + \kappa_z^2(m, \omega), \qquad m = 1, \ldots, M \quad (4)$$

where κ_z is the vertical wave number and $c(z)$ the depth-dependent sound speed profile as before.

If the continuous source spectrum is decomposed into a sampled or discrete spectrum using a periodic impulse (frequency) sampler [14], then it follows that

$$S_s(\omega) = \triangle\omega \sum_q S(\omega) \delta(\omega - \omega_q) = \triangle\omega \sum_q S(\omega_q) \delta(\omega - \omega_q) \quad (5)$$

from the sampling properties of the Fourier transform [14]. Therefore a broadband signal spectrum can be decomposed into a set of narrowband components assuming an impulse sampled spectrum at each spectral bin, $q\triangle\omega$.

Utilizing this property in Eq. 5 and extracting just the qth source frequency, we have that

$$p(r, z, \omega_q) = \mathcal{G}(r, z, \omega) S_s(\omega_q) \quad (6)$$

where $S_s(\omega_q)$ can be interpreted as a single narrowband impulse at ω_q with amplitude, $a_q = \triangle\omega |S(\omega_q)|$. Suppose that the broadband source is assumed to be bandlimited and sampled (as before), that is,

$$S_s(\omega) \qquad \omega_1 \leq \omega \leq \omega_Q \quad (7)$$

then

$$S_s(\omega) = \triangle\omega \sum_{q=1}^{Q} S(\omega_q) \delta(\omega - \omega_q) \quad (8)$$

Thus, the normal-mode solution to the Helmholtz equation for the broadband source problem can be decomposed into a series of narrowband solutions $\{\omega_q\}; q = 1, \cdots, Q$, that is,

$$p(r, z, \omega_q) = \sum_{m=1}^{M_q} a_q \mathcal{H}_o \left(\kappa_r(m, q) r \right) \phi_m(z_s, \omega_q) \phi_m(z, \omega_q)$$
$$\quad (9)$$

B. State-Space Propagators

It is well-known [11], [12] that the state-space propagators for the narrowband pressure-field can be obtained from the depth relationship and the broadband extensions discussed. Following the development in Ref[12], we define the modal state vector for the mth mode, at frequency ω_q, as

$$\Phi_m(z, \omega_q) = \begin{bmatrix} \phi_m(z, \omega_q) \\ \frac{d}{dz} \phi_m(z, \omega_q) \end{bmatrix}, \quad m = 1, \cdots, M_q. \quad (10)$$

Equation 5 can now be written in state-space form as

$$\frac{d}{dz} \Phi_m(z, \omega_q) = \mathbf{A}_m(z, \omega_q) \Phi_m(z, \omega_q), \quad (11)$$

where

$$\mathbf{A}_m(z, \omega_q) = \begin{bmatrix} 0 & 1 \\ -\kappa_z^2(m, q) & 0 \end{bmatrix}, \quad m = 1, \cdots, M_q. \quad (12)$$

Expanding over the M_q modes at each narrowband frequency ω_q component, we have that

$$\frac{d}{dz} \mathbf{\Phi}(z, \omega_q) = \mathbf{A}(z, \omega_q) \mathbf{\Phi}(z, \omega_q) \quad (13)$$

with $\mathbf{A}(z, \omega_q) = \text{diag}[\mathbf{A}_1(z, \omega_q)| \cdots |\mathbf{A}_{M_q}(z, \omega_q)]$ the single frequency state vector defined by

$$\mathbf{\Phi}^T(z, \omega_q) := \left[\Phi_1^T(z, \omega_q) \cdots \Phi_{M_q}^T(z, \omega_q) \right] \quad (14)$$

for $\mathbf{\Phi}^T(z, \omega_q) \in R^{2M_q \times 1}$.

Suppose we further assume an L-element vertical sensor array, then $z \rightarrow z_\ell, \ell = 1, \cdots, L$ and therefore, the pressure-field at the array for the qth temporal frequency of Eq. 9 becomes

$$p(r, z_\ell, \omega_q) = \sum_{m=1}^{M_q} \beta_m(r, z_\ell, \omega_q) \phi_m(z_\ell, \omega_q) \quad (15)$$

where $\beta_m(r, z_\ell, \omega_q) := a_q \mathcal{H}_o \left(\kappa_r(m, q) r \right) \phi_m(z_s, \omega_q)$ is the mth-modal coefficient at the qth temporal frequency.

Thus, the broadband pressure-field at the ℓth-sensor is simply

$$p(r, z_\ell) = \frac{1}{Q} \sum_{q=1}^{Q} p(r, z_\ell, \omega_q) \qquad (16)$$

which corresponds to incoherently summing all of the narrowband solutions over the discrete spectral lines (bins) [7], [12]. Thus, the amount of spectral energy "seen" at the ℓth-sensor in the qth-spectral bin ($\omega_q = q\triangle\omega$) is defined by $p(r, z_\ell, \omega_q)$ and therefore the total energy seen by the array in the qth-bin is given by the set of sensor outputs, $\bar{P}(\omega_q) := \{p(z_1, \omega_q), \cdots p(z_L, \omega_q)\}$.

Therefore, if we sample the spatial pressure-field with a vertical line array of hydrophones, then at the ℓth hydrophone we have

$$p(z_\ell, \omega_q) = \mathbf{C}_q^T(r_s, z_s, \omega_q)\boldsymbol{\Phi}(z_\ell, \omega_q) \qquad (17)$$

with the *measurement matrix* given by

$$\mathbf{C}_q^T(r_s, z_s, \omega_q) = \begin{bmatrix} \beta_1(r_s, z_s, \omega_q) & 0 | \cdots | \beta_{M_q}(r_s, z_s, \omega_q) & 0 \end{bmatrix} \qquad (18)$$

where $\mathbf{C}_q^T \in \mathcal{R}^{1 \times M_q}$, $r \to r_s$ to give

$$\beta_m(r_s, z_s, \omega_q) := \mathcal{H}_o(\kappa_r(m, q)r_s)\phi_m(z_s, \omega_q) \qquad (19)$$

These relations constitute the *state equation* and *measurement equation*, respectively, to be applied for temporal frequency ω_q. This model is only valid for a single temporal frequency, ω_q. Extending it to the broadband case, we see from Eqs. 11 and 12 that as the temporal frequency (ω_q) changes, the corresponding numbers of modes (M_q) and horizontal wave numbers ($\kappa_r(m, q)$) at each frequency also change. Thus, the state-space propagator for the broadband case must allow for this frequency dependence.

Bearing this issue in mind, we now define the overall *broadband state-space propagator* as

$$\frac{d}{dz}\boldsymbol{\Phi}(z, \Omega) = \mathbf{A}(z, \Omega)\boldsymbol{\Phi}(z, \Omega) \qquad (20)$$

with

$$\mathbf{A}(z, \Omega) = diag\left[\mathbf{A}(z, \omega_1) \cdots \mathbf{A}(z, \omega_Q)\right], \quad \mathbf{A}(z, \omega) \in \mathbf{R}^{2M \times 2M} \qquad (21)$$

and

$$\boldsymbol{\Phi}^T(z_\ell, \Omega) := \begin{bmatrix} \boldsymbol{\Phi}^T(z_\ell, \omega_1) & \cdots & \boldsymbol{\Phi}^T(z_\ell, \omega_Q) \end{bmatrix} \qquad (22)$$

for $\boldsymbol{\Phi}^T(z_\ell, \Omega) \in \mathbf{R}^{2M \times 1}$ and (as before),

$$\mathbf{A}_m(z, \omega_q) = \begin{bmatrix} 0 & 1 \\ -\kappa_z^2(m, q) & 0 \end{bmatrix} \qquad (23)$$

with $q = 1, \ldots, Q \leq N$; $m = 1, \ldots, M_q$. The parameter M, the total number of single frequency modes, is given by $M = \sum_1^Q M_q$. Note that we use the parameter "Ω" to signify the entire set of discrete temporal frequencies, $\{\omega_q\}, q = 1, \cdots, Q$.

Finally, the overall broadband pressure measurement equation takes the following form where we choose to make a *incoherent average* over temporal frequencies, that is,

$$p(z_\ell, \Omega) = \mathbf{C}(r_s, z_s, \Omega)\boldsymbol{\Phi}(z_\ell, \Omega) \qquad (24)$$

with

$$\mathbf{C}(r_s, z_s, \Omega) = \frac{1}{Q} \begin{bmatrix} \mathbf{C}_1^T(r_s, z_s, \omega_1) & \cdots & \mathbf{C}_Q^T(r_s, z_s, \omega_Q) \end{bmatrix} \qquad (25)$$

for $\mathbf{C} \in \mathcal{R}^{1 \times M}$.

C. Stochastic State-Space Propagators

The stochastic nature of this broadband ocean acoustic problem requires the introduction of *uncertainties* not just in the overall propagation of noise and disturbances, but also in the parametric uncertainties in the underlying modal models. For these reasons we incorporate additive stochastic processes to capture the noisy, varying shallow ocean medium, that is, discretizing Eq. 20 using central differences [23] gives

$$\begin{aligned} \boldsymbol{\Phi}(z_\ell, \Omega) &= \mathbf{A}(z_{\ell-1}, \Omega)\boldsymbol{\Phi}_m(z_{\ell-1}, \Omega) + \mathbf{w}(z_{\ell-1}, \Omega) \\ p(z_\ell, \Omega) &= \mathbf{C}(r_s, z_s, \Omega)\boldsymbol{\Phi}(z_\ell, \Omega) + v(z_\ell, \Omega) \end{aligned} \qquad (26)$$

where w is an independent, uncorrelated, zero-mean process with spectral covariance $R_{ww}(z_\ell, \Omega)$, $v \sim \mathcal{N}(0, R_{vv}(z_\ell, \Omega))$ and $\boldsymbol{\Phi}(z_0, \Omega) \sim \mathcal{N}(\overline{\boldsymbol{\Phi}}(z_0, \Omega), R_{\phi\phi}(z_o, \Omega))$.

This completes the fundamental state-space representation providing the required ocean acoustic models for the subsequent processor.

III. BROADBAND BAYESIAN PROCESSOR

In order to develop the Bayesian processor, we must cast our problem into a probabilistic framework under these assumptions; therefore, our sequence of pressure-field measurements at each sensor are Fourier transformed to yield a discrete set of frequencies $\{\omega_q\}, q = 1, \cdots, Q$ and the set of broadband sensor measurements, $\{\mathbf{p}(z_\ell, \Omega)\}, \ell = 1, \cdots, L$. Note that the window length of the Fourier transform is determined by the temporal correlation time of the measurement (source) to assure the independence of the frequency samples. For our pressure-field/modal function estimation problem, we define the underlying *broadband pressure-field/modal function enhancement problem* as:

GIVEN a set of noisy broadband pressure-field measurements, $P_\ell := \{\mathbf{p}(z_1, \Omega), \cdots, \mathbf{p}(z_\ell, \Omega)\}$ and the underlying stochastic model of Eq. 26, FIND the best estimate of the posterior distribution, $\text{Pr}\left[\boldsymbol{\Phi}(z_\ell, \Omega)|P_\ell\right]$ to infer the enhanced broadband modal functions, $\hat{\boldsymbol{\Phi}}(z_\ell, \Omega)$ and pressure-field, $\hat{\mathbf{p}}(z_\ell, \Omega)$.

The Bayesian solution to this problem can be obtained by estimating the *a posteriori* distribution as follows.

$$\Pr\left[\mathbf{\Phi}(z_\ell,\Omega)|P_\ell\right] = \frac{\Pr\left[\mathbf{\Phi}(z_\ell,\Omega),P_\ell\right]}{\Pr\left[P_\ell\right]} \qquad (27)$$

but from Bayes' rule we have that

$$\Pr\left[\mathbf{\Phi}(z_\ell,\Omega)|P_\ell\right] =$$

$$\frac{\Pr\left[\mathbf{p}(z_\ell,\Omega)|\mathbf{\Phi}(z_\ell,\Omega),P_{\ell-1}\right] \times \Pr\left[\mathbf{\Phi}(z_\ell,\Omega),P_{\ell-1}\right]}{\Pr\left[\mathbf{p}(z_\ell,\Omega)|P_{\ell-1}\right] \times \Pr\left[P_{\ell-1}\right]}$$
$$(28)$$

Now expanding the second term in the numerator again using Bayes' rule, we have

$$\Pr\left[\mathbf{\Phi}(z_\ell,\Omega),P_{\ell-1}\right] = \Pr\left[\mathbf{\Phi}(z_{\ell-1},\Omega)|P_{\ell-1}\right] \times \Pr\left[P_{\ell-1}\right] \qquad (29)$$

Substituting this relation in the previous equation and cancelling like terms we obtain the required expression for the posteriori density

$$\Pr\left[\mathbf{\Phi}(z_\ell,\Omega)|P_\ell\right] =$$

$$\frac{\Pr\left[\mathbf{p}(z_\ell,\Omega)|\mathbf{\Phi}(z_\ell,\Omega),P_{\ell-1}\right] \times \Pr\left[\mathbf{\Phi}(z_\ell,\Omega)|P_{\ell-1}\right]}{\Pr\left[\mathbf{p}(z_\ell,\Omega)|P_{\ell-1}\right]}$$
$$(30)$$

A. Broadband Particle Filters

One approach to estimate the required posterior distribution from noisy broadband measurements is to develop the so-called particle filter (*PF*) [16]. A *particle filter* provides an estimate of an empirical probability mass function (*PMF*) that approximates the desired posterior distribution such that statistical inferences can easily be performed and statistics extracted directly. As expected the computational burden of the *PF* is much higher that of other processors, since it must provide an estimate of the underlying state posterior distribution component-by-component at *each* z_ℓ-step along with the fact that the number of samples to characterize the distribution is equal to the number of particles.

$$\hat{\Pr}[\Phi(z_\ell,\Omega)|P_\ell] = \sum_{i=1}^{N_p} \mathcal{W}_i(z_\ell,\Omega)\delta\big(\Phi(z_\ell,\Omega)-\Phi_i(z_\ell,\Omega)\big)$$
$$(31)$$

$$\begin{aligned}
\mathcal{W}_i(z_\ell,\Omega) &\propto \hat{\Pr}[\Phi_i(z_\ell,\Omega)|P_\ell] \\
&\text{is the estimated weights at depth } z_\ell \\
\Phi_i(z_\ell,\Omega) &\quad \text{is the } i\text{-th particle at depth } z_\ell \\
\hat{\Pr}[\cdot] &\quad \text{is the estimated empirical distribution} \\
P_\ell &\quad \text{is the set of batch pressure-field} \\
&\quad \text{measurements}, P_\ell = \{p(r_s,z_1)\cdots p(r_s,z_\ell)\}
\end{aligned}$$

Thus, we see that once the underlying posterior is available, the estimates of important statistics can be extracted directly. For instance, the maximum a posteriori (*MAP*) estimate is simply found by locating a particular particle $\hat{\phi}_i(z_\ell)$ corresponding to the maximum of the *PMF*, that is,

$$\hat{\Phi}_i^{MAP}(z,\Omega) = \max_i \ \hat{\Pr}[\Phi_i(z_\ell,\Omega)|P_\ell] \qquad (32)$$

while the conditional mean or equivalently the minimum mean-squared error (*MMSE*) estimate is calculated by integrating the posterior as:

$$\hat{\Phi}_i^{MMSE}(z,\Omega) \approx \frac{1}{N_p} \sum_{i=1}^{N_p} \mathcal{W}_i(z_\ell,\Omega)\Phi_i(z_\ell,\Omega) \qquad (33)$$

The generic solution of the particle filter is based on deriving the associated weighting function. A sampling or equivalently importance distribution $\mathcal{I}\big(\Phi(z_\ell,\Omega)|P_\ell\big)$ is selected first, then the weight is determined by the ratio of the desired posterior to this choice [16]

$$\mathcal{W}\big(z_\ell,\Omega\big) := \frac{\Pr\left[\Phi(z_\ell,\Omega)|P_\ell\right]}{\mathcal{I}\big(\Phi(z_\ell,\Omega)|P_\ell\big)}$$

which can be expanded using Bayes' rule to give the "sequential" generic weight

$$\mathcal{W}\big(z_\ell,\Omega\big) := \mathcal{W}\big(z_{\ell-1},\Omega\big) \times$$
$$\frac{\Pr\left[p(z_\ell,\Omega)|\Phi(z_\ell,\Omega)\right] \times \Pr\left[\Phi(z_\ell,\Omega)|\Phi(z_{\ell-1},\Omega)\right]}{\mathcal{I}\big(\Phi(z_\ell,\Omega)|P_\ell\big)}$$
$$(34)$$

where the numerator is simply the product of the usual *likelihood* and the so-called state *transition* probability.

B. Bootstrap Particle Filters

There are a variety of *PF* algorithms available based on the choice of the importance distribution [16], [17]. Perhaps the simplest is the *bootstrap* technique [22]. The *PF* design for our problem using the bootstrap approach selects the state transition probability as its importance distribution, that is,

$$\mathcal{I}\big(\Phi(z_\ell,\Omega)|P_\ell\big) := \Pr\left[\Phi(z_\ell,\Omega)|\Phi(z_{\ell-1},\Omega)\right] \qquad (35)$$

leading to the weighting function

$$\mathcal{W}\big(z_\ell,\Omega\big) = \mathcal{W}\big(z_{\ell-1},\Omega\big) \times \Pr\left[p(z_\ell,\Omega)|\Phi(z_\ell,\Omega)\right] \qquad (36)$$

which is simply the *likelihood distribution*.

For the bootstrap implementation, we need only draw noise samples from the state distribution and use the dynamic model of Eq. 26 to generate the set of particles, $\{\Phi_i(z_\ell,|\Omega)\}$ for each $i = 1, \cdots, N_p$.

The likelihood, on the other hand, is determined from the pressure-field measurement model, that is, for each mode we have

$$p_i(z_\ell, \Omega) = \mathbf{C}(r_s, z_s, \Omega)\mathbf{\Phi}_i(z_\ell, \Omega) + v(z_\ell, \Omega) \qquad (37)$$

and therefore the scalar likelihood (assuming Gaussian measurement noise) is

$$\Pr[p(z_\ell, \Omega)|\Phi(z_\ell, \Omega)] = \frac{1}{\sqrt{2\pi R_{vv}(\Omega)}} \times$$
$$\exp\left\{ -\frac{1}{2R_{vv}(\Omega)}\Big(p(z_\ell, \Omega) - \mathbf{C}(r_s, z_s, \Omega)\mathbf{\Phi}_i(z_\ell, \Omega)\Big)^2 \right\}$$
$$(38)$$

Thus, we estimate the posterior distribution using a sequential Monte Carlo approach and construct a *bootstrap particle filter* [16] using the following steps:

- Initialize: $\Phi_i(0, \Omega), w_{z_\ell, \Omega} \sim \mathcal{N}(0, R_{ww}(\Omega)), W_i(0, \Omega) = 1/N_p; i = 1, \cdots, N_p$;

- State Transition: $\Phi_i(z_\ell, \Omega) = \mathbf{A}(z_{\ell-1}, \Omega)\Phi_i(z_{\ell-1}, \Omega) + \mathbf{w}_i(z_{\ell-1}, \Omega)$;

- Likelihood Probability: $\Pr[p(z_\ell, \Omega)|\Phi(z_\ell, \Omega)]$ of Eq. 38;

- Weights: $W_i(z_\ell, \Omega) = W_i(z_{\ell-1}, \Omega) \times \Pr[p(z_\ell, \Omega)|\Phi(z_\ell, \Omega)]$;

- Normalize: $\mathcal{W}_i(z_\ell, \Omega) = \frac{W_i(z_\ell, \Omega)}{\sum_{i=1}^{N_p} W_i(z_\ell, \Omega)}$;

- Resample: $\tilde{\Phi}_i(z_\ell, \Omega) \Rightarrow \Phi_i(z_\ell, \Omega)$;

- Posterior: $\hat{\Pr}[\Phi_i(z_\ell, \Omega)|P_\ell] = \sum_{i=1}^{N_p} \mathcal{W}_i(z_\ell, \Omega)\delta(\Phi(z_\ell, \Omega) - \Phi_i(z_\ell, \Omega))$; and

- *MAP* Estimate: $\hat{\Phi}_i^{MAP}(z, \Omega) = \max_i \hat{\Pr}[\Phi_i(z_\ell, \Omega)|P_\ell]$;

- *MMSE* Estimate: $\hat{\Phi}_i^{MMSE}(z, \Omega) = \frac{1}{N_p}\sum_{i=1}^{N_p} \mathcal{W}_i(z_\ell, \Omega)\Phi_i(z_\ell, \Omega)$

IV. BROADBAND BAYESIAN DESIGN FOR A SHALLOW OCEAN

In this section we discuss the application of the Bayesian processor to data synthesized by a broadband normal-mode model using the state-space forward propagator and the underlying stochastic representation of the previous section. In order to develop the propagator we *first* must define the shallow water boundary value problem and solve it to obtain the parameters required for the processor.

It is important to realize that the state-space "forward" propagators do *not* offer an alternative solution to the Helmholtz equation, but rather use the parameters *from* the boundary

value solution to obtain a set of initial conditions/parameters for propagator construction. Therefore, in order to implement the processor modal parameters (e.g. wave numbers, sound speed, etc.) of the ocean medium under investigation must be provided or else jointly estimated in a parametrically adaptive scheme [23], [24]. This is not uncommon in any of the model-based approaches [14]. Modal parameters, $\Theta(z_\ell, \Omega)$, are obtained using, for example, SNAP [19], KRACKEN [20], SAFARI [21] providing initial parameter estimates and inputs to the Bayesian processor for a variety of applications.

Consider a basic shallow water ocean channel assuming a flat bottom, range independent three layer environment with a channel depth of $100m$, a sediment depth of $2.5m$ and a subbottom. A vertical line array of 100-sensors with spacing of $\triangle z = 1m$ spans the entire water column and a broadband source of unit amplitude and $50Hz$ bandwidth ranging from $50 - 100Hz$ in $10Hz$ increments is located at a depth of $50m$ and a range of $10Km$ from the array. The sound speed profile in the water column and the sediment are sketched in the figure and specified along with the other problem parameters in Table I. SNAP, a normal-mode propagation simulator [19] is applied to solve this shallow water problem and executed over the set of discrete temporal source frequencies shown in Table I as well. This boundary value problem was solved using SNAP and the results at each narrowband frequency are shown in Table II below. We note from the Table (as expected) that as the temporal frequency increases, the number of modes increases.

Next we design the state-space propagator. The parameters obtained from SNAP are now used to construct the broadband state-space and measurement models of Eq. 26. Here we use the set of horizontal wave numbers, $\{\kappa(m, q)\}, m = 1, \cdots, M_q; q = 1, \cdots, Q$, and sound speed, $\{c(z_\ell)\}$, to implement the state-space models along with the corresponding modal function values, $\{\Phi(z_s, \omega_q)\}$, as well as the Hankel functions, $\{\mathcal{H}_o(\kappa(m, q)r_s)\}$ to construct the measurement models (modal coefficients).

The final set of parameters for our simulation are the modal and measurement noise covariance matrices required by the stochastic model. Both are specified by the input (modal) and output (measurement) signal-to-noise ratios (*SNR*) defined by:

$$SNR_{in} := \frac{\text{Cov}\Big(\mathbf{\Phi}_m(z_\ell, \omega_q)\Big)_{m,m}}{R_{ww}(m, m)}, \quad m = 1, \cdots, M_q$$
$$SNR_{out} := \frac{\mathbf{C}(r, z, \Omega)\text{Cov}\Big(\mathbf{\Phi}_m(z, \Omega)\Big)\mathbf{C}^T(r, z, \Omega)}{R_{vv}}$$
$$(39)$$

With this information in hand, the stochastic simulation was performed at $SNR_{in} = 10dB$ (noise free) and $SNR_{out} = 0db$. A typical realization is shown in Figs. 4 and 6 where the uncertain set of modal functions at each frequency are depicted in Fig. 4 along with the measured pressure-field ($0dB$)in Fig. 6. The Bayesian processor is then designed using the identical

set of parameters used in the shallow water simulation. We can consider this simulation as a bound on the best that can be achieved.

Table I.   Shallow Ocean Simulation Parameters (SNAP)

Problem Parameters			
Parameter	Water	Sediment	Bottom
$Depth(m)$	100	2.5	-
$Density(G/cm^3$	1	1.8	1.84
$Attenuation(dB/NL)$	0.13	0.15	0.0
Source			
$Frequency(Hz)$	50-100		
$Range(Km)$	10		
$Depth(m)$	50		
SoundSpeed			
0.0	1503.0	1597.95	1597.95
0.1		1597.95	
0.2		1522.58	
1.6		1537.65	
2.5		1552.73	
5.0	1503.1		
10.0	1503.2		
15.0	1503.3		
20.0	1503.4		
100.0	1504.7		

In modal/pressure-field estimation it is important to realize the overall design philosophy. First, a key issue is that the error (residual or innovation) sequence that is the difference between measured and predicted pressure-fields should be zero-mean and uncorrelated (white), if all correlated (modal) information has been captured by the processor, that is,

$$\epsilon(z_\ell, \Omega) := p(z_\ell, \Omega) - \hat{p}(z_\ell, \Omega) \qquad (40)$$

Thus, as a starting point designs are *not* considered "tuned" unless this condition is satisfied; therefore, the free parameters in the processor (usually initial conditions and process/measurement noise vectors/matrices) are adjusted until this condition is achieved. Once satisfied, then and only then can the state (modal function) and measurement (pressure-field) estimates along with their associated covariances be considered viable. This is a consistent metric applied throughout the statistical signal processing community [14]. To test that the residual sequence is zero-mean, white we use the sample statistics (whiteness test), the weighted sum-squared residual statistic as metrics [16]. The usual whiteness/zero-mean tests, that is, testing that 95% of the sample (normalized) residual correlations lie within the bounds or equivalently 5% or less fall outside the bounds.

Table II.   Shallow Ocean Boundary Solutions

SNAP Parameters			
Freq(Hz)	Mode No.	Modal Coeff.	Wave No.
50	1	0.122	0.207386
	2	-0.070	0.202489
60	1	0.125	0.249339
	2	-0.063	0.245106
	3	-0.097	0.237639
70	1	0.127	0.291255
	2	-0.057	0.287522
	3	-0.108	0.280907
80	1	0.129	0.333144
	2	-0.052	0.329802
	3	-0.113	0.323913
	4	0.087	0.315526
90	1	0.130	0.375015
	2	-0.047	0.371985
	3	-0.117	0.366685
	4	0.084	0.358997
100	1	0.131	0.416871
	2	-0.043	0.414098
	3	-0.119	0.409278
	4	0.079	0.402279
	5	0.081	0.393399

When data are nonstationary then a more reliable statistic to use is the *weighted sum-squared residual* (*WSSR*) which is a measure of the overall global estimation performance for the Bayesian processor, determining the "whiteness" of the residual error (innovation) sequence [14]. It uses this sequence to test whiteness by requiring that the constructed decision function lies below a specified threshold. If the *WSSR* statistic does lie beneath the calculated threshold, then theoretically, the estimator is tuned and said to converge. Here the window is designed to slide through the residual data and estimate its whiteness. Thus, overall performance of the processor can be assessed by analyzing the statistical properties of the residual errors, which is essentially the approach we take in this test for the broadband processor design on synthesized data performance as well as the calculated *MSE* estimates.

The results of the Bayesian design are shown in Fig. 1 where we see the enhanced pressure-field and the corresponding innovations sequence as a function of depth. Note from Eq. 26 that the modal estimates $\hat{\boldsymbol{\Phi}}(z_\ell, \omega_q)$ along with the measurement model at each temporal frequency, $C^T(r_s, z_s, \omega_q)$ are used to construct the enhanced pressure-field, $\hat{\mathbf{p}}(z_\ell, \omega_q)$ at each temporal frequency along with the corresponding innovation. The sequence test zero-mean and white as demonstrated by the whiteness tests and the corresponding *WSSR* statistic lying below the threshold in Fig. 2. To complete the performance analysis, we observe the posterior pressure-field distribution predicted by the *PF* at each depth in Fig. 3. Clearly this distribution is not unimodal, but both *MAP/MMSE* inferences "track" the mean (true) pressure-field quite well as confirmed by the zero-mean/whiteness tests of the residual errors.

Thus, we have achieved a "broadband" design. Note that the enhanced pressure-field estimates (*MAP*, *MMSE*), are

governed by the model of Eq. 26 and indicates precisely what the array is sampling at a particular frequency.

The estimated modal functions extracted from the noisy pressure-field measurements are viable estimates and shown in Fig. 4 bounding the estimates we could hope to achieve for this type of data. Here we observe both *MAP* and *MMSE* estimates inferred from the predicted posterior modal distributions. The results are quite reasonable. In Fig. 4 we observe the enhanced modal functions corresponding to the two (2) modes at $50Hz$ and three (3) at $60Hz$. The other estimated modes (from the noisy data) at $70Hz$ (3 modes), $80Hz$ (4 modes), $90Hz$ (4 modes) and $100Hz$ (5 modes) are also shown.

Modal function tracking of the broadband data presents another aspect of the *PF* estimates. The *RMS MSE* estimates for each of the modal functions are shown in Table III where we observe small mode tracking errors again indicating a well-tuned Bayesian processor. The estimated posterior distribution (slices) at depths of $z = 1m$, $25m$, $50m$, $75m$, $100m$ are shown in Fig. 5. Each slice is a cut (probability vs particles at a given depth) of the posterior surface for each particular mode. Figure 9 provides a glimpse of how the actual particles "coalesce" around distribution peaks (regions of highest importance) from the posterior modal distributions provided by the *PF* varying at given depths. This completes the application of the broadband Bayesian processor designed to enhance the pressure-field surface and extract the corresponding modal functions.

V. SUMMARY

In this paper we have developed a sequential Bayesian solution to the broadband pressure-field enhancement and modal function tracking (enhancement) problem. Modeling a shallow ocean environment by a normal-mode propagator, we developed a broadband Bayesian solution. We showed how each of the corresponding temporal frequency bands lead to an underlying state-space structure which is eventually used in the development of a forward propagator for simulation and the resulting processor for enhancement. We developed a shallow ocean simulation using SNAP [19] to solve the associated boundary value problem first and supplied these parameters to implement the Bayesian processor. We showed the results of the design demonstrating the capability of such an approach.
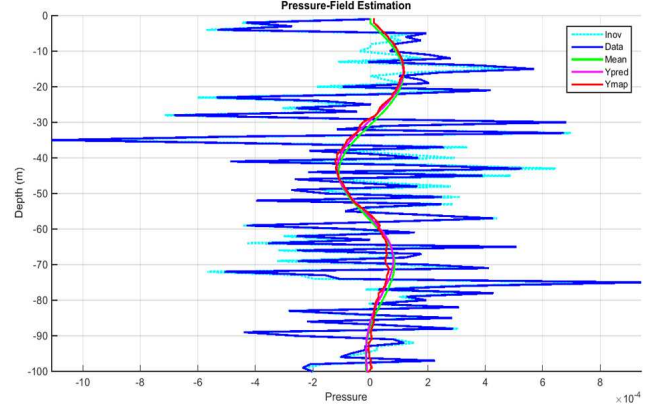
Acknowledgments

Fig. 1. Broadband shallow water environment pressure-field enhancement: Raw data (SNR=0dB), particle filter estimates (MAP, MMSE) and residual errors (innovations).
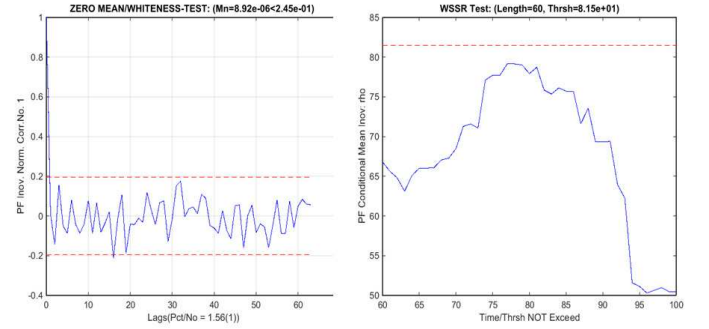

Fig. 2. Broadband Bayesian processor residual error (innovation) sequence zero-mean/whiteness test results: (1.56% out; $9 \times 10^{-6} < 0.245$) and *WSSR* below threshold.
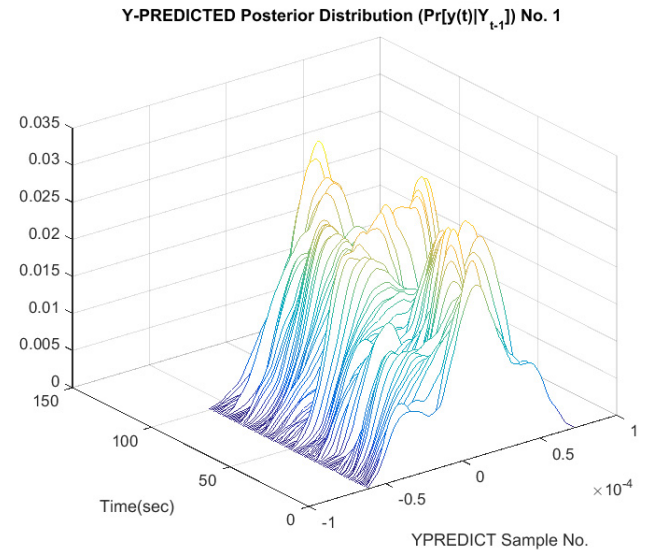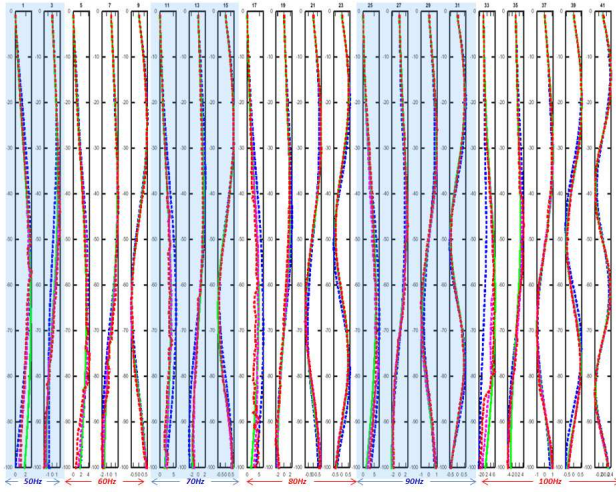

Fig. 3. Predicted pressure-field posterior distribution.

Fig. 4. Broadband shallow water environment mode tracking: Particle filter modal function (21) estimation (MAP, MMSE).

Fig. 5. Posterior modal distributions (21) at $1m$, $25m$, $50m$, $75m$ and $100m$.

Table III. Mode Tracking Estimation

PF Performance		
Frequency(Hz)	Mode No.	*MSE*
50	1	0.03244469
	2	0.06390081
60	1	0.02057981
	2	0.02262004
	3	0.00725442
70	1	0.1173259
	2	0.07374703
	3	0.01009431
80	1	0.1376377
	2	0.03716592
	3	0.01584901
	4	0.01425319
90	1	0.1104416
	2	0.02928122
	3	0.01980555
	4	0.00782307
100	1	0.3176212
	2	0.06316895
	3	0.01011795
	4	0.01544138
	5	0.00450539

REFERENCES

[1] A. Parvulescu. "Signal detection in a multipath medium by MESS processing," *J. Acoust. Soc. Am.*, **29**, 223-228, 1965.

[2] C. S. Clay and H. Medwin *Acoustical Oceanography* New York: John Wiley, 1977.

[3] C. S. Clay. "Optimum time domain signal transmission and source localization in a waveguide," *J. Acoust. Soc. Am.*, **81**, 660-664, 1987.

[4] S. Li and C. S Clay. "Optimum time domain signal transmission and source localization in a waveguide: experiments in an ideal wedge waveguide," *J. Acoust. Soc. Am.*, **82**, 1409-1417, 1987.

[5] R. K. Brienzo and W. Hodgkiss, "Broadband matched-field processing". *J. Acoust. Soc. Am.*, **94**, 1409-1417, 1994.
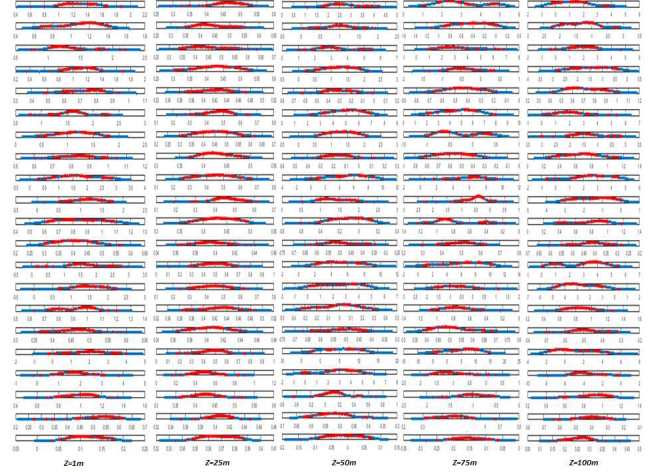
[6] J. P. Hermand and W. I. Roderick, "Acoustic model-based matched-filter processing for fading time dispersive ocean channels: Theory and experiment," IEEE J. Oceanic Eng., **18**, 447-465, 1993.

[7] A. B. Baggeroer, W. A. Kuperman, and H. Schmidt, "Matched-field processing: source localization in correlated noise as an optimum parameter estimation problem," *J. Acoust. Soc. Am*, **83**, (2), 571-587, 1988.

[8] T. C. Yang. "Broadband source localization and signature estimation," *J. Acoust. Soc. Am*, **93**, (4), 1797-1806, 1993.

[9] I. T. Lu, H. Y. Chen, and P. Voltz. "A matched-mode processing technique for localizing a transient source in the time domain," *J. Acoust. Soc. Am*, **93**, (3), 1365-1373, 1993.

[10] A. M. Richardson, and L. W. Nolte, "A posteriori probability source localization in an uncertain sound speed, deep ocean environment," *J. Acoust. Soc. Am.*, **89**, (6), 2280-2284, 1991

[11] J. V. Candy and E. J. Sullivan. "Ocean acoustic signal processing: a model-based approach." *J. Acoust. Soc. Am.*, **92**, (12), 3185-3201, 1992.

[12] J. V. Candy and E. J. Sullivan. "Broadband model-based processing for shallow ocean environments." *J. Acoust. Soc. Am.*, **104**, (1), 275-287, 1998.

[13] F. B. Jensen, W. A. Kuperman, M. B. Porter and H. Schmidt, *Computational Ocean Acoustics*. New York:AIP Press, 1994.

[14] J. V. Candy, *Model-Based Signal Processing*. Hoboken, N.J.:John Wiley/IEEE Press, 2006.

[15] W. Kuperman and F. Ingenito, "Spatial correlation of surface generated noise in a stratified ocean," *J. Acoust. Soc. Am.*, **67**, (6), 1988-1996, 1980.

[16] J. V. Candy, *Bayesian Signal Processing: Classical, Modern and Particle Filtering Methods*. Hoboken, N.J.:Wiley/IEEE Press, 2009.

[17] B. Ristic, S. Arulampalam and N. Gordon, *Beyond the Kalman Filter: Particle Filters for Tracking Applications*, Boston: Artech House, 2004.

[18] P. Djuric, J. Kotecha, J. Zhang, Y. Huang, T. Ghirmai, M. Bugallo and J. Miguez, "Particle Filtering," *IEEE Signal Proc. Mag.* vol. **20**, No. 5, 19-38, 2007.

[19] F. B. Jensen, and M.C. Ferla, "SNAP: the SACLANTCEN normal-mode acoustic propagation model," *SACLANTCEN Report*, **SM-121**, SACLANT Undersea Research Centre, La Spezia, Italy, 1982.

[20] M. B. Porter, "The KRACKEN normal mode program," *Report SM-245*, Italy: SACLANTCEN, 1991.

[21] H. Schmidt, "SAFARI: Seismo-acoustic fast field algorithm for range independent environments," *Report SM-245*, Italy: SACLANTCEN, 1987.

[22] J. V. Candy, "Bootstrap Particle Filtering: performance of a passive synthetic aperture in an uncertain ocean environment," *IEEE Signal Proc. Mag.* vol. **24**, No. 4, 73-85, 2007.

[23] J. V. Candy, "Adaptive particle filtering for mode tracking: a shallow ocean environment," *Proc. OCEANS'11*, Santander, Spain, 2011.

[24] J. V. Candy, "An adaptive particle filtering approach to tracking modes is a varying shallow ocean environment," *Proc. OCEANS'11*, Hawaii (Big Island), Hawaii, 2011.